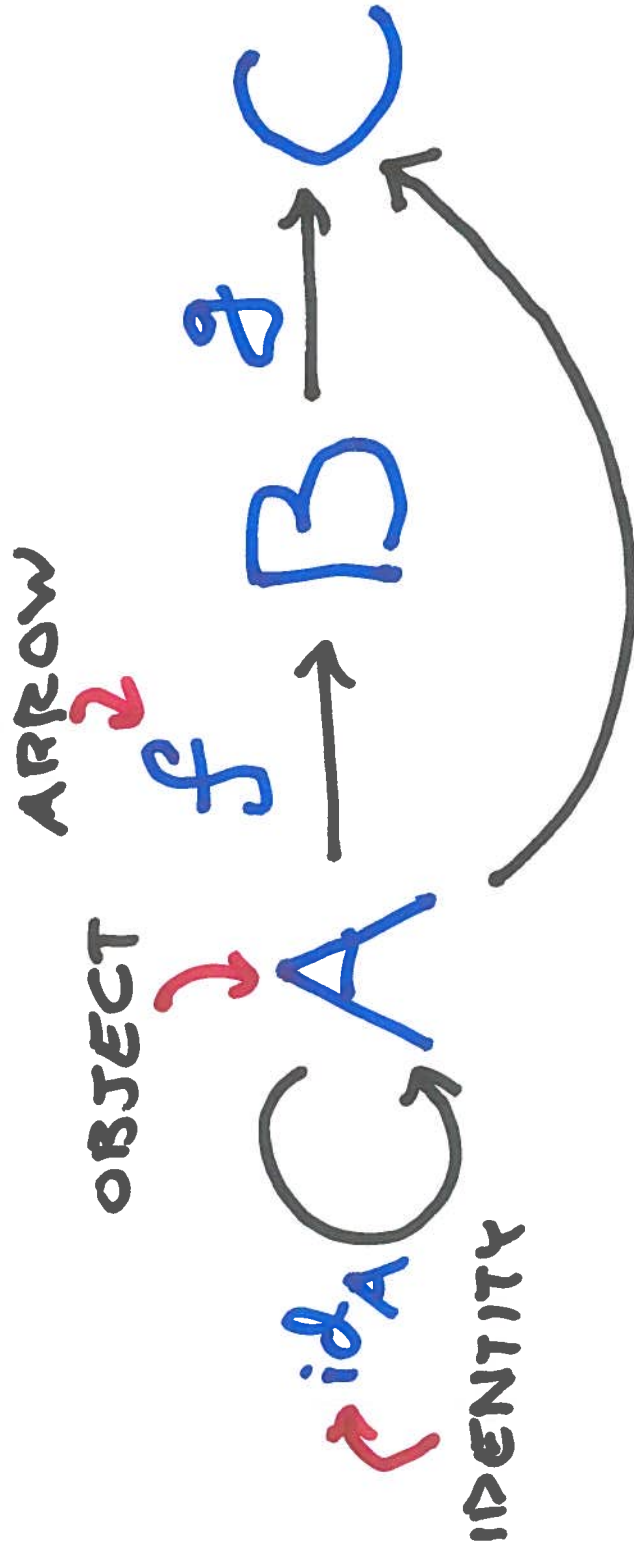


CATEGORIES for the WORKING PROGRAMMER

Philip Wadler
University of Edinburgh
Joy of Code, 17 June 2016

CATEGORY $\mathcal{C}$



$f;g$

COMPOSITION

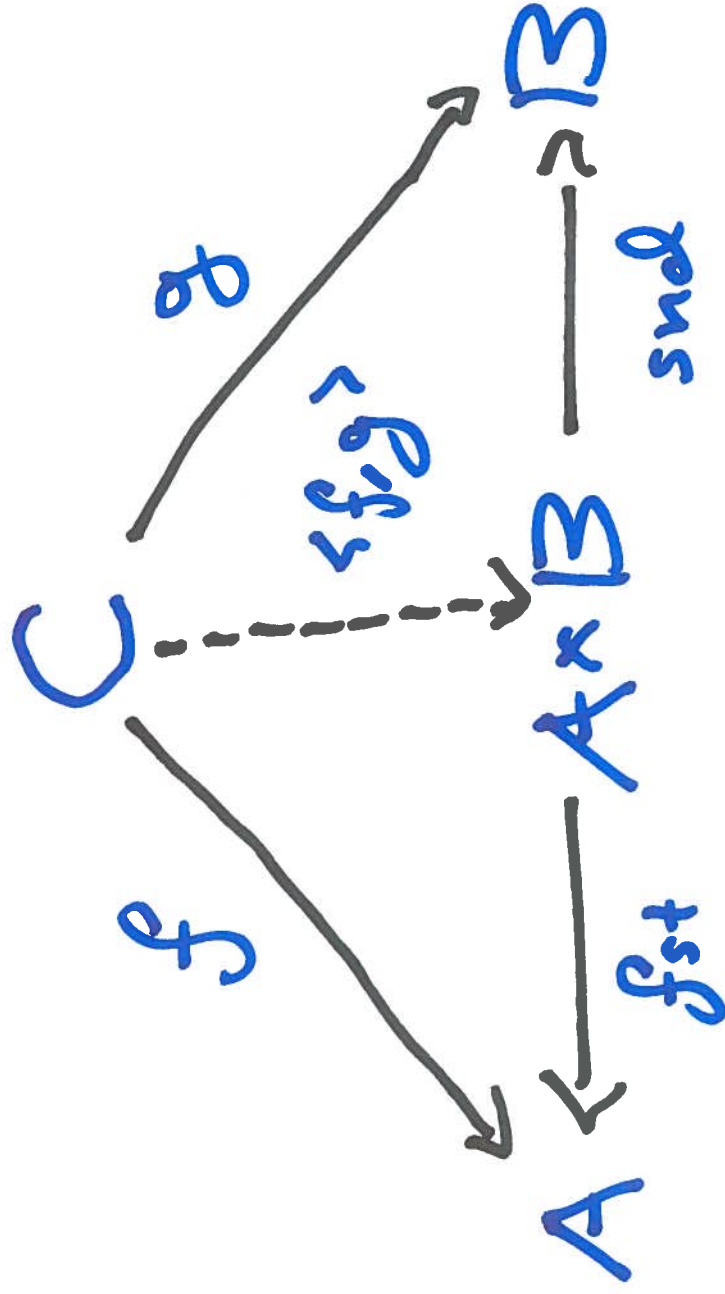
$$\text{id}_A; f = f; \text{id}_B$$

$$(f; g); h = f; (g; h)$$

$\mathcal{C}(A, B)$

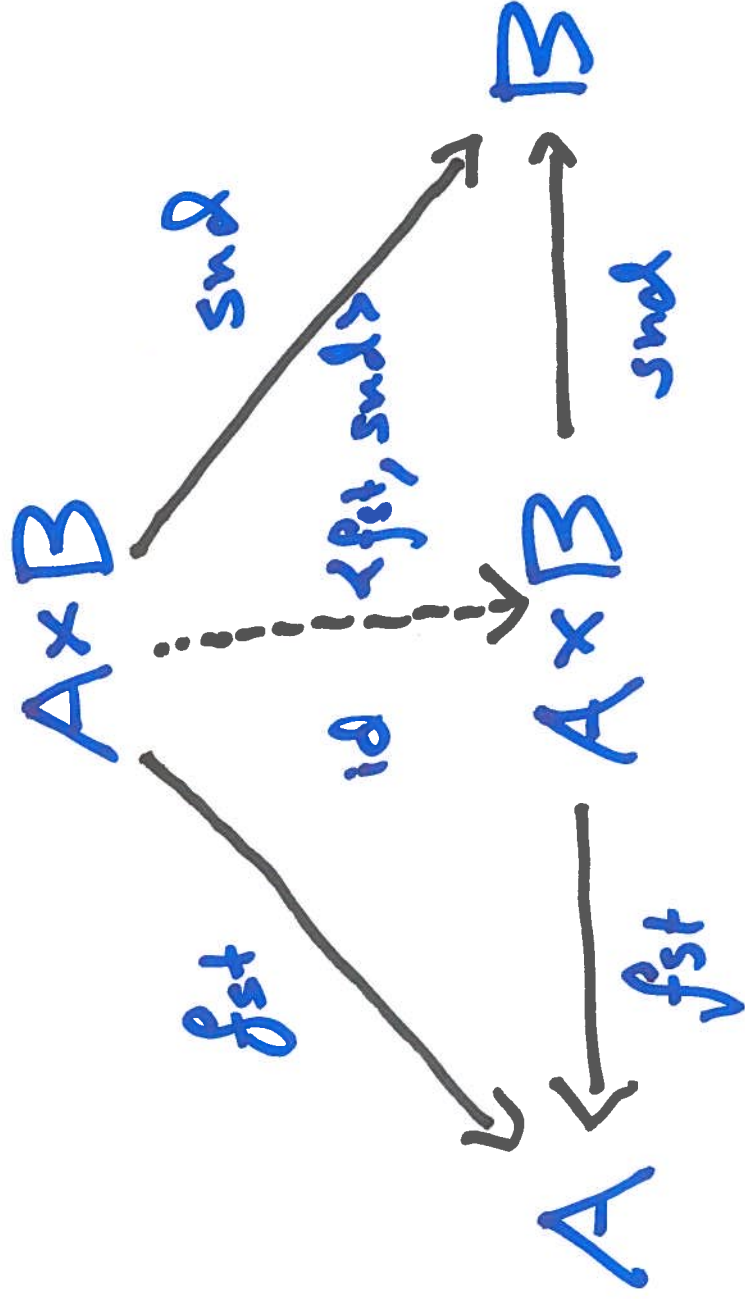
$$= \{ A \xrightarrow{f} B \}$$

PRODUCT



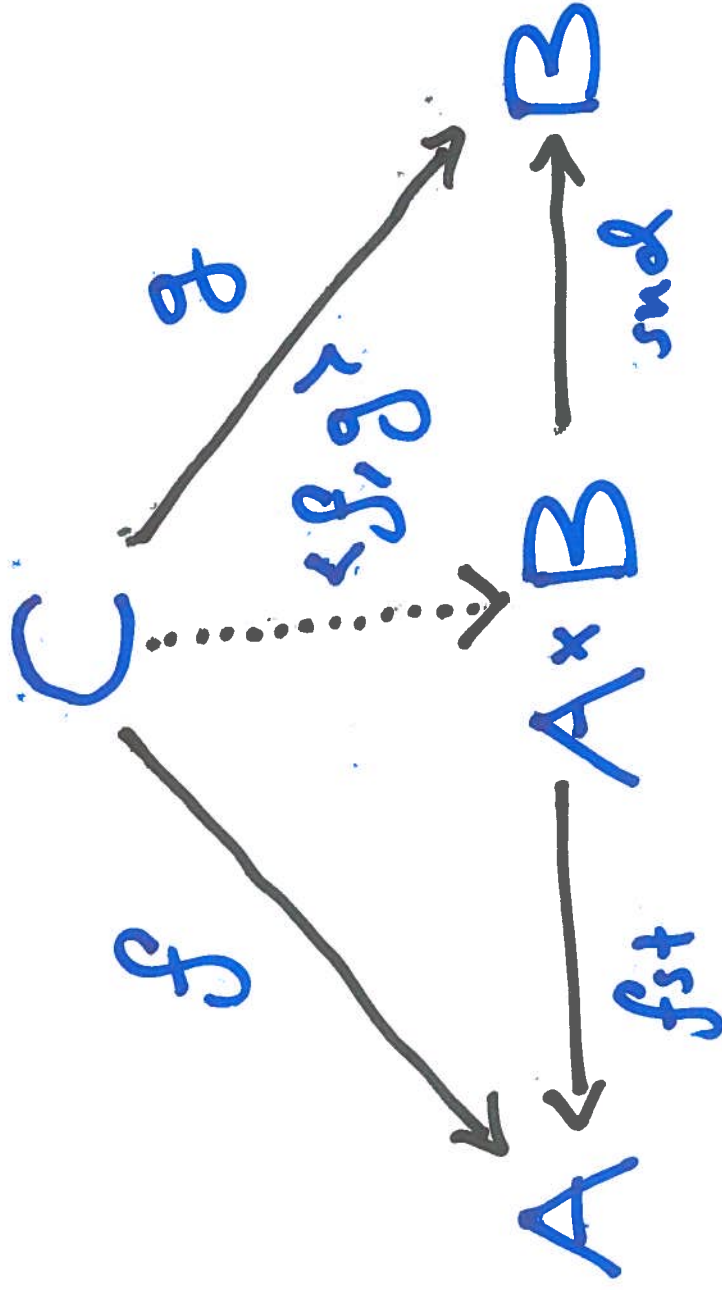
$$\langle -, - \rangle : \mathcal{C}(C, A) \times \mathcal{C}(C, B) \cong \mathcal{C}(C, A \times B)$$

PRODUCT



$$\langle \text{fst}, \text{snd} \rangle = \text{id}$$

PRODUCT



$$\begin{aligned}
 \langle f, g \rangle; f_{st} &= f \\
 \langle f, g \rangle; g_{nd} &= g \\
 \Rightarrow h; f_{st} &= f \quad \& \quad h; g_{nd} = g \\
 \Rightarrow h &= \langle f, g \rangle
 \end{aligned}$$

PRODUCT

$$\begin{array}{c} \Gamma \vdash M : A \\ \Gamma \vdash N : B \\ \hline \Gamma \vdash (M, N) : A \times B \end{array} \quad x\text{-I}$$

$$\begin{array}{c} \Gamma \vdash L : A \times B \\ \hline \Gamma \vdash \text{fst } L : A \end{array} \quad x\text{-E}$$

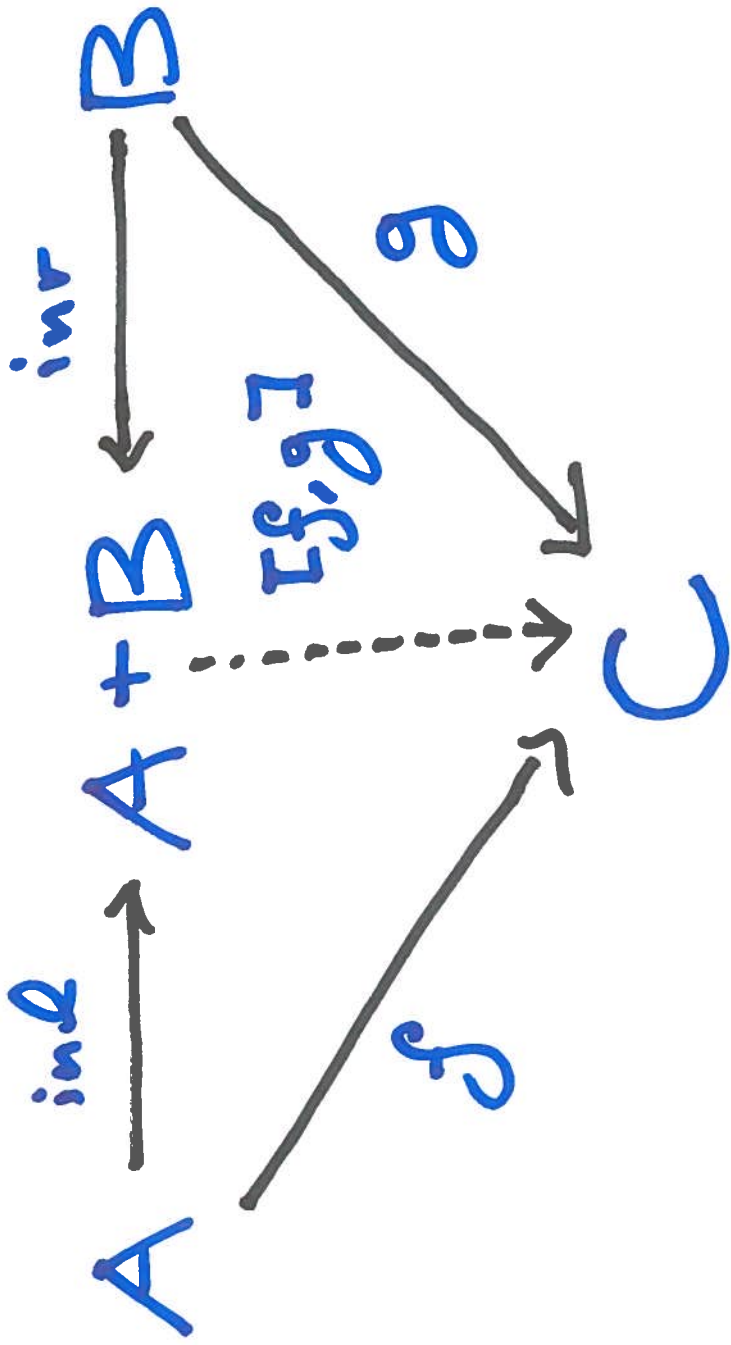
$$\begin{array}{c} \Gamma \vdash L : A \times B \\ \hline \Gamma \vdash \text{snd } L : B \end{array} \quad x\text{-E}$$

$$\begin{array}{c} \Gamma \xrightarrow{f} A \\ \Gamma \xrightarrow{g} B \\ \hline \Gamma \xrightarrow{f \cdot g} A \times B \end{array}$$

$$\begin{array}{c} \Gamma \xrightarrow{h} A \times B \\ \hline \Gamma \xrightarrow{h} A \times B \xrightarrow{\text{fst}} A \end{array}$$

$$\begin{array}{c} \Gamma \xrightarrow{h} A \times B \\ \hline \Gamma \xrightarrow{h} A \times B \xrightarrow{\text{snd}} B \end{array}$$

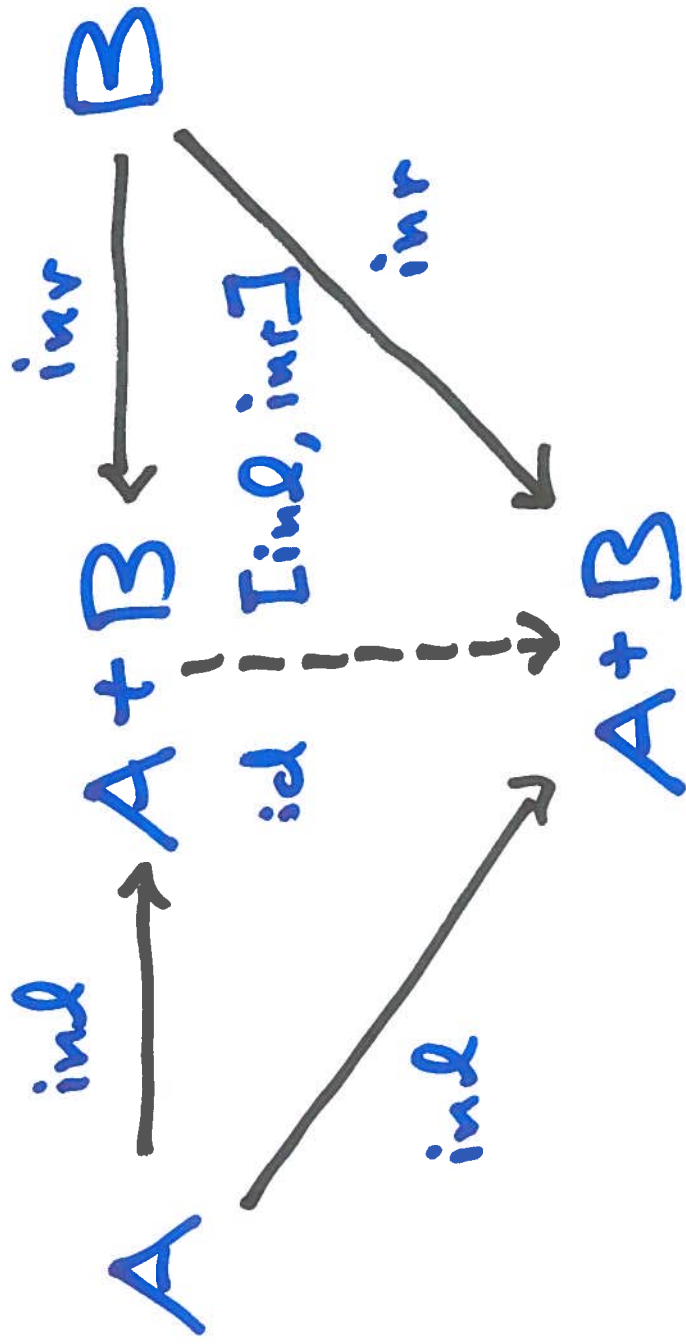
SUM



$$\begin{aligned} \text{ind}; [f, g] &= f \\ \text{inv}; [f, g] &= g \end{aligned}$$

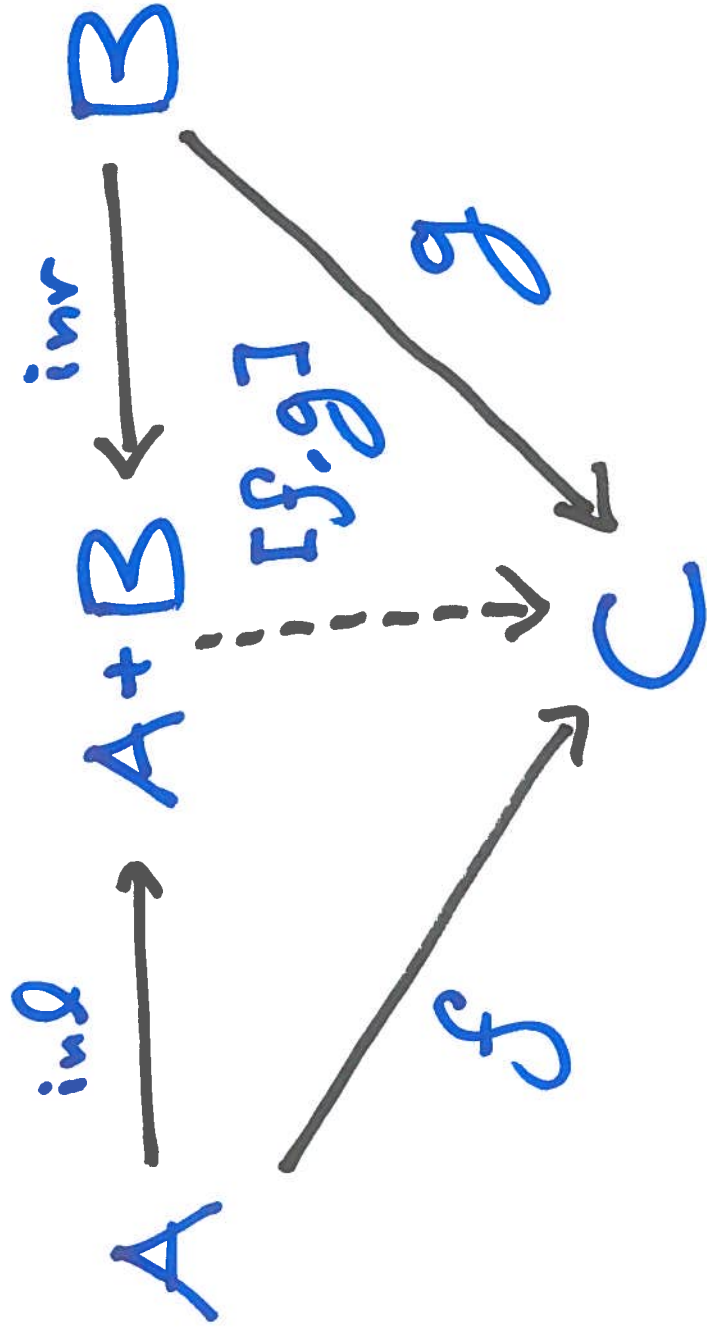
$$\text{ind}; h = f \ \& \ \text{inv}; h = g \Rightarrow h = [f, g]$$

SUM



$$[\text{inl}, \text{inr}] = \text{id}$$

SUM



$$[-, -] : \mathcal{C}(A, C) \times \mathcal{C}(B, C) \rightarrow \mathcal{C}(A+B, C)$$

SUM

$$\frac{\Gamma \vdash M : A}{\vdash I \quad \Gamma \vdash \text{ind } M : A+B}$$

$$\frac{\Gamma \vdash N : B}{\vdash I \quad \Gamma \vdash \text{inv } N : A+B}$$

$$\frac{\begin{array}{l} \Gamma \vdash L : A+B \\ \Gamma, x:A \vdash P : C \\ \Gamma, y:B \vdash Q : C \end{array}}{\vdash E \quad \Gamma \vdash (\text{case } L \text{ of } P \rightarrow Q) : C}$$

$\begin{array}{cc} \text{ind } x \rightarrow & \text{inv } y \rightarrow \\ \text{ind } x \rightarrow & \text{inv } y \rightarrow \end{array}$

$$\frac{\Gamma \vdash A}{\Gamma \vdash f \rightarrow A \xrightarrow{\text{ind}} A+B}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash g \rightarrow B \xrightarrow{\text{inv}} A+B}$$

$$\frac{\begin{array}{l} \Gamma \vdash h \rightarrow A+B \\ \Gamma \times A \xrightarrow{f} C \\ \Gamma \times B \xrightarrow{g} C \end{array}}{\Gamma \xrightarrow{\text{cid}, h} \Gamma \times (A+B)} \quad \begin{array}{c} \text{cid}, g, I \\ \hline \Gamma \times A + \Gamma \times B \xrightarrow{\text{cid}, g, I} C \end{array}$$

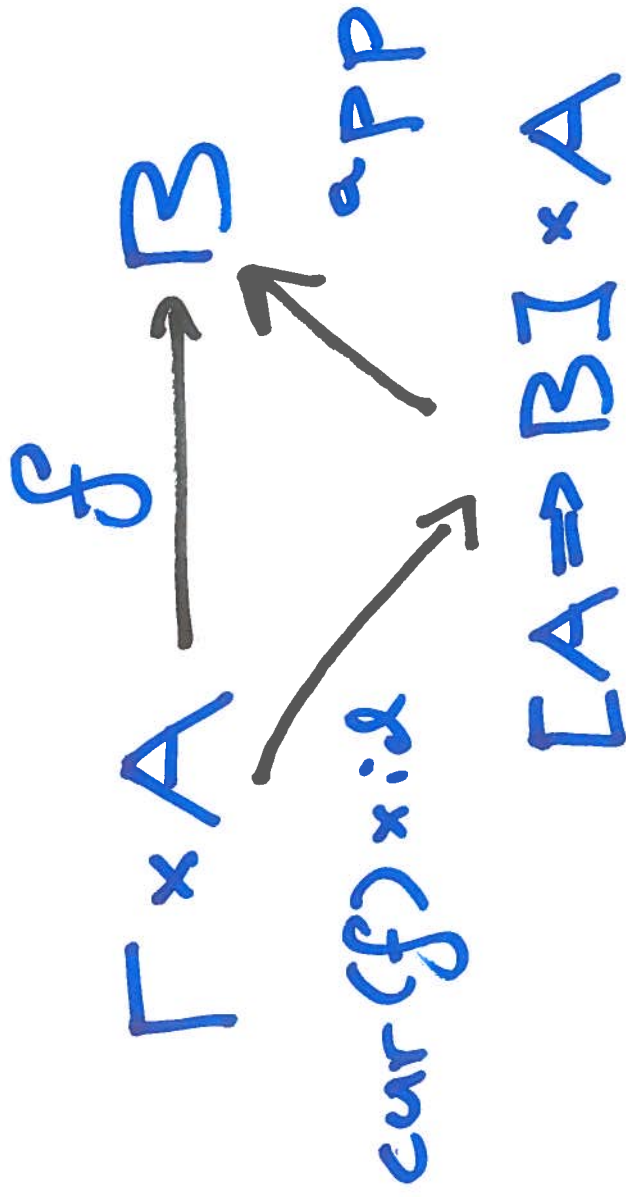
DISTRIBUTIVE

$\langle \text{fst}; \text{ind}, \text{fst}; \text{inr} \rangle, \langle \text{snd}, \text{snd} \rangle \rangle$

$$(A+C) * (B+C) \hat{=} (A+B) * C$$

$\langle \text{fst}; \langle \text{cur}(\text{ind}), \text{cur}(\text{inr}) \rangle \rangle, \text{snd} \rangle; \text{app}$

EXPONENTIAL



$$\mathcal{C}(\Gamma * A, B) \cong \mathcal{C}(\Gamma, [A \Rightarrow B])$$

VARIABLES

$$\frac{}{x:A \vdash x:A}$$

$$\frac{\Gamma \vdash N:B}{\Gamma, x:A \vdash N:B}$$

$$\frac{\Gamma \vdash N:B}{x:A, \Gamma \vdash N:B}$$

$$\frac{}{A \xrightarrow{id} A}$$

$$\frac{\Gamma \xrightarrow{\eta} N}{\Gamma \times A \xrightarrow{\xi} \Gamma \xrightarrow{\eta} N}$$

$$\frac{\Gamma \xrightarrow{\eta} N}{A \times \Gamma \xrightarrow{\eta} \Gamma \xrightarrow{\eta} N}$$

EXPONENTIAL

$$\rightarrow I \quad \frac{\Gamma, x:A \vdash N:B}{\Gamma \vdash (\lambda x:A. N) : [A \Rightarrow B]}$$

$$\Gamma \vdash L : [A \Rightarrow B]$$

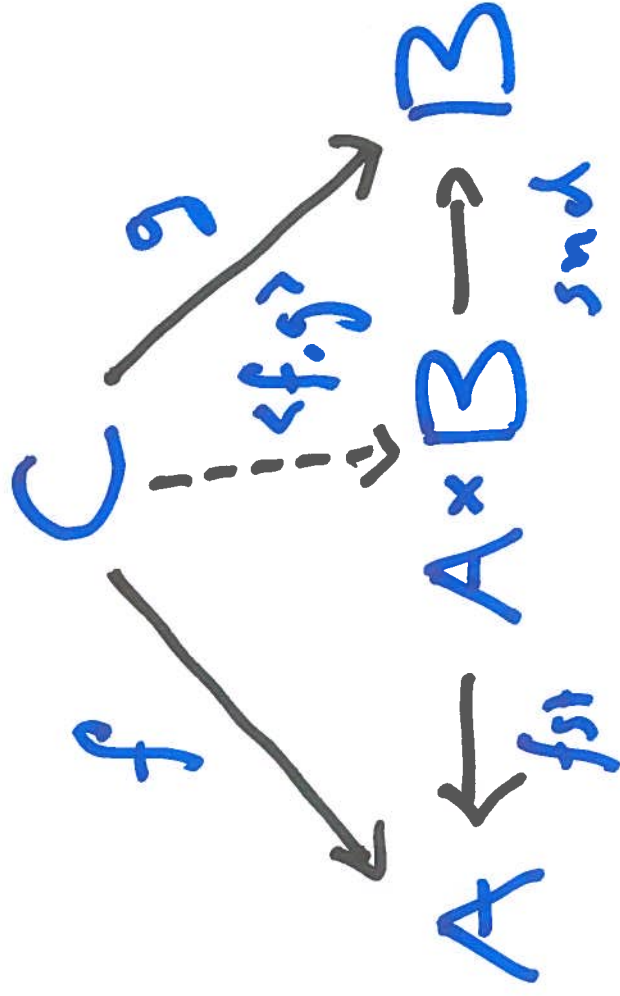
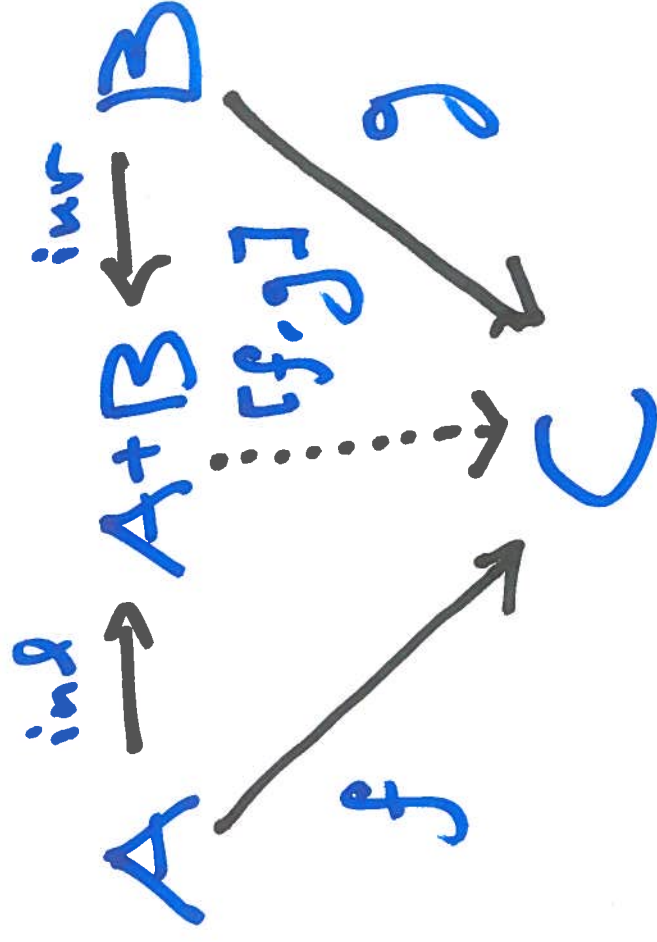
$$\Gamma \vdash M : A$$

$$\rightarrow E \quad \frac{\Gamma \vdash L : [A \Rightarrow B] \quad \Gamma \vdash M : A}{\Gamma \vdash L M : B}$$

$$\frac{\Gamma * A \xrightarrow{f} B}{\Gamma \xrightarrow{\text{cur}(f)} [A \Rightarrow B]}$$

$$\frac{\begin{array}{c} \Gamma \xrightarrow{f} [A \Rightarrow B] \\ \Gamma \xrightarrow{g} A \end{array}}{\Gamma \xrightarrow{\{f, g\}} [A \Rightarrow B] * A \xrightarrow{\text{app}} B}$$

PRODUCTS & SUMS



ISOMORPHISMS

$$\mathcal{C}(C, A) \times \mathcal{C}(C, B) \cong \mathcal{C}(C, A^*B)$$

$$\mathcal{C}(A, C) \times \mathcal{C}(B, C) \cong \mathcal{C}(A^*B, C)$$

$$\mathcal{C}(C^*A, B) \cong \mathcal{C}(C, [A \Rightarrow B])$$

HIGH SCHOOL

$$A^C \times B^C = (A \times B)^C$$

$$C^A \times C^B = C^{(A+B)}$$

$$B^{(C \times A)} = (B^A)^C$$